113 Class Problems: Normal Subgroups and Isomorphism Theorems

1. Let p be a prime number and H be a group. Prove the following:

   If $\phi : \mathbb{Z}/p\mathbb{Z} \to H$ is a homomorphism, then either ϕ (i.e $Im(\phi) = \{e_H\}$) or ϕ is injective.

   Hint: Consider the kernel of ϕ.

   Solution:

$\ker\phi \subset \mathbb{Z}/p\mathbb{Z}$ a subgroup.

$\Rightarrow |\ker\phi| \,\big|\, p \Rightarrow |\ker\phi| = 1 \Rightarrow \phi$ injective

or

$|\ker\phi| = p \Rightarrow Im\,\phi = \{e_H\}$

2. Let $H = \{e, (123), (132)\} \subset Sym_4$. By giving an example, show that the following binary operation is not well-defined:

$$\begin{array}{rcl} Sym_4/H \times Sym_4/H & \to & Sym_4/H \\ (xH, yH) & \mapsto & xyH \end{array}$$

   Solutions:

Let $x_1 = y_1 = (14)$, $x_2 = y_2 = (14)(123) = (1234)$

$\Rightarrow x_1H = x_2H$

$y_1H = y_2H$

$x_1y_1H = eH$

$x_2y_2H = (1234)^2H = (13)(24)H$

$(13)(24) \notin H \Rightarrow eH \neq (13)(24)H$

$\Rightarrow x_1y_1H \neq x_2y_2H$

3. Let G, H be two finite groups. Prove the following:

   If there exists a non-trivial $\phi : G \to H$ (ie. $Im(\phi) \neq \{e_H\}$) then $HCF(|G|, |H|) > 1$.

   Hint: Think about the First Isomorphism Theorem.

   Solutions:

$\phi : G \to H$ non-trivial $\Rightarrow |Im\,\phi| > 1$

Lagrange $\Rightarrow |Im\,\phi| \mid |H|$

F.I.T $\Rightarrow |Im\,\phi| \mid |G|$

$\Rightarrow HCF(|G|, |H|) > 1$

4. Let N be a normal subgroup of G. If $(G : N) = 7$ determine all subgroups $H \subset G$ such that $N \subset H$.

   Hint: Think about the Third Isomorphism Theorem.

   Solutions:

$N \triangleleft G \Rightarrow G/N$ is a group of size 7

7 prime $\xRightarrow{\text{Lagrange}}$ Only subgroups of G/N are $\{eN\}$ and G/N

Third Isomorphism Theorem $\Rightarrow$ Only intermediate subgroups of G and N are N and G.